Log-convexity and log-concavity for series in gamma ratios and applications

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Abstract. Polynomial sequence $\{P_m\}_{m\geq 0}$ is q-logarithmically concave if $P_m^2 - P_{m+1}P_{m-1}$ is a polynomial with nonnegative coefficients for any $m \geq 1$. We introduce an analogue of this notion for formal power series whose coefficients are nonnegative continuous functions of parameter. Four types of such power series are considered where parameter dependence is expressed by a ratio of gamma functions. We prove six theorems stating various forms of q-logarithmic concavity and convexity of these series. The main motivating examples for these investigations are hypergeometric functions. In the last section of the paper we present new inequalities for the Kummer function, the ratio of the Gauss functions and the generalized hypergeometric function obtained as direct applications of the general theorems.

Keywords: *Gamma function, log-concavity, log-convexity, q-log-concavity, Wright log-concavity, Turán inequality, Kummer function, generalized hypergeometric function*

MSC2010: 26A51, 33C20, 33C15, 33C05

1. Introduction. We adopt standard notation $\mathbb{N}$ for the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}$ will stand for reals and $\mathbb{R}_+$ for nonnegative reals. The gamma function $\Gamma(x)$ was introduced by Leonard Euler who also demonstrated that its second logarithmic derivative is positive for positive values of x. In modern language this means that $\Gamma(x)$ is logarithmically convex (i.e. its logarithm is a convex function). A sum of log-convex functions can be shown to be log-convex using Hölder inequality or a theorem of Montel [21, Theorem 1.4.5.2]. Additivity implies then that the (finite or infinite) sum $f(\mu;x) := \sum f_k\Gamma(\mu+k)x^k$ is logarithmically convex function of μ for fixed $x \geq 0$ once the coefficients f_k are assumed to be nonnegative. It is not difficult to see that much more is true [17, Theorem 2]: the formal power series $f(\mu;x)f(\mu+\alpha+\beta;x) - f(\mu+\alpha;x)f(\mu+\beta;x)$ has nonnegative coefficients at all powers of x if $\alpha,\beta \geq 0$. In [14] we considered a similar problem for the series $g(\mu;x) := \sum g_k\{\Gamma(\mu+k)\}^{-1}x^k$. Here each term is log-concave function of μ, so that lack of additivity of logarithmic concavity does not allow to draw any immediate conclusion about the sum. We have demonstrated, however, that the sequence $\{g(\mu;x)\}_{\mu\in\mathbb{N}}$ is log-concave for fixed $x > 0$ if the sequence of coefficients $\{g_k\}_{k\in\mathbb{N}}$ is log-concave. Moreover, in this case $g(\mu;x)g(\mu+\alpha+\beta;x) - g(\mu+\alpha;x)g(\mu+\beta;x)$ has nonnegative coefficients at all powers of x if $\alpha,\beta \in \mathbb{N}$. The two sums above can be generalized naturally to series in product ratios of gamma function having the form (3) below. Several known questions in financial mathematics [5, 6], multidimensional statistics [30], probability [24] and special functions [3, 4, 12] reduce to or depend on log-convexity or log-concavity of special cases of such generalized series. Similar coefficient-wise positivity of product differences is also important in combinatorics. The following definition is attributed to Richard Stanley [28, p.795]. A sequence of polynomials $\{P_m(x)\}_{m\geq 0}$ is said to be q-log-concave if

$$P_m(x)^2 - P_{m+1}(x)P_{m-1}(x)$$

is a polynomial with nonnegative coefficients for any $m \geq 1$. It is strongly q-log-concave if

$$P_m(x)P_n(x) - P_{m+1}(x)P_{n-1}(x)$$

is a polynomial with nonnegative coefficients for all $m \geq n \geq 1$. The latter notion was introduced by Sagan [28]. Many sequences of combinatorial polynomials especially those related to q-calculus possess these properties (see [8, 28] for details and references). We will need extensions of these notions to families of formal power series. To be consistent with the standard definitions of log-concavity and Wright log-concavity [22, Chapter I.4], [26, Section 1.1] and to make our formulations more compact, we found it reasonable to change the combinatorial terminology slightly. Suppose

$$f(\mu; x) = \sum_{k=0}^{\infty} f_k(\mu)x^k \tag{1}$$

is a formal power series with nonnegative coefficients which depend continuously on a nonnegative parameter μ.

Definition. The family $\{f(\mu; x)\}_{\mu\geq 0}$ is **Wright q-log-concave** if formal power series

$$\phi_\mu(\alpha, \beta; x) := f(\mu + \alpha; x)f(\mu + \beta; x) - f(\mu; x)f(\mu + \alpha + \beta; x) \tag{2}$$

has nonnegative coefficients at all powers of x for all $\mu, \alpha, \beta \geq 0$. If this property only holds for $\alpha \in \mathbb{N}$ and all $\mu, \beta \geq 0$ we will say that $\{f(\mu; x)\}_{\mu\geq 0}$ is **discrete Wright q-log-concave**. Finally, $\{f(\mu; x)\}_{\mu\geq 0}$ is **discrete q-log-concave** if $\phi_\mu(\alpha, \beta; x)$ has nonnegative coefficients at all powers of x for $\alpha \in \mathbb{N}$, $\beta \geq \alpha - 1$ and all $\mu \geq 0$.

If each function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is associated with the family of formal power series $\{f(\mu; x)\}_{\mu\geq 0}$ with constant term $f_0 = f(\mu)$ and zero coefficients at all positive powers of x, the above definitions become consistent with the following standard terminology: $\mu \to f(\mu)$ is called Wright log-concave on $\mathbb{R}_+$ if $f(\mu + \alpha)f(\mu + \beta) \geq f(\mu)f(\mu + \alpha + \beta)$ for all $\mu, \alpha, \beta \geq 0$ [22, Chapter I.4], [26, Definition 1.13]; it is discrete Wright log-concave on $\mathbb{R}_+$ if the above inequality holds for $\alpha \in \mathbb{N}$ and all $\mu, \beta \geq 0$ and discrete log-concave if it holds for $\alpha \in \mathbb{N}$, $\beta \geq \alpha - 1$ and $\mu \geq 0$ [14]. For continuous functions Wright log-concavity is equivalent to log-concavity (i.e. concavity of the logarithm). Discrete Wright log-concavity implies discrete log-concavity but not vice versa (see details in [14]). All above definitions also apply if we substitute "concave" by "convex", "non-negative" by "non-positive" and reverse the sign of all inequalities. In the theory of special functions discrete log-concavity and log-convexity are also frequently referred to as "Turán type inequalities" following the classical result of Paul Turán for Legendre polynomials [31]: $[P_n(x)]^2 > P_{n-1}(x)P_{n+1}(x)$, $-1 < x < 1$. Note, however, that the sequence $\{P_n(x)\}_{n\geq 0}$ is not q-log-concave. General Wright convex functions attracted a lot of attention recently (see, for instance, [11, 18] and references therein) following a fundamental result of Ng [23].

If $f : \mathbb{N}_0 \to \mathbb{R}_+$ is a sequence, then discrete log-concavity reduces to inequality $f_k^2 \geq f_{k-1}f_{k+1}$, $k \in \mathbb{N}$. We will additionally require that the sequence $\{f_k\}_{k=0}^{\infty}$ is non-trivial and has no internal zeros, i.e. $f_N = 0$ implies either $f_{N+i} = 0$ for all $i \in \mathbb{N}_0$ or $f_{N-i} = 0$ for $i = 0, 1, \ldots, N$. Such sequences are also known as PF_2 (Pólya frequency sub two) or doubly positive [13]. Clearly, if f is (Wright) log-concave then $1/f$ is (Wright) log-convex. Notwithstanding the simplicity of this relation, several important properties of log-concavity and log-convexity differ. As we already mentioned above, log-convexity is preserved under addition while log-concavity is not. Further, log-convexity is a stronger property than convexity whereas log-concavity is weaker than concavity. Further properties of log-convex and log-concave functions can be found, for instance, in [19, 3E, 16D, 18B], [24, Chapter 2] and [26, Chapter 13].

The questions considered in [14, 17] and in this paper are particular cases of the following general problem: under what conditions on a nonnegative sequence $\{f_k\}$ and the numbers a_i, b_j the series

$$f(\mu;x)=\sum_{k=0}^{\infty}f_k\frac{\prod_{i=1}^{n}\Gamma(a_i+\mu+\varepsilon_i k)}{\prod_{j=1}^{m}\Gamma(b_j+\mu+\varepsilon_{n+j}k)}x^k \tag{3}$$

is (discrete, Wright) q-log-concave or q-log-convex? Here ε_r can take values 1 or 0. In particular, if the ratio f_{k+1}/f_k is a rational function of k the series in (3) is hypergeometric (possibly times some gamma functions) and μ represents parameter shift [2, Chapter 2].

In [17] the authors treated the cases of (3) with $n=1$, $m=0$, $\varepsilon_1=1$; $n=m=1$, $\varepsilon_1=1$, $\varepsilon_2=0$, $a_1=b_1$; and $n=m=1$, $\varepsilon_1=0$, $\varepsilon_2=1$, $a_1=b_1$. In [14] we handled $n=0$, $m=1$, $\varepsilon_1=1$. In this paper we treat the following cases:

(a) $n=m=2$, $\varepsilon_1=1$, $\varepsilon_2=0$, $\varepsilon_3=0$, $\varepsilon_4=1$, $a_1=b_1$, $a_2=b_2$;
(b) $n=m=1$, $\varepsilon_1=1$, $\varepsilon_2=1$;
(c) $n=2$, $m=1$, $\varepsilon_1=1$, $\varepsilon_2=0$, $\varepsilon_3=1$, $a_2=b_1$;
(d) $n=1$, $m=2$, $\varepsilon_1=1$, $\varepsilon_2=0$, $\varepsilon_3=1$, $a_1=b_1$.

For small values of n and m considered in (a)-(d) it is easy to determine the conditions for each term in (3) to be log-convex. By additivity we can then assert the log-convexity of $\mu\to f(\mu;x)$ for fixed $x\ge 0$, but not q-log-convexity of any type, i.e. non-positivity of the Taylor coefficients of $\phi_\mu(\alpha,\beta;x)$ defined by (2). If the terms in (3) are log-concave even the verification of log-concavity of $\mu\to f(\mu;x)$ for fixed $x\ge 0$ becomes non-trivial. In this paper we verify Wright q-log-convexity and discrete q-log-concavity for the family of power series defined in (3) under restrictions (a)-(d). Our results will imply then that either $x\to\phi_\mu(\alpha,\beta;x)$ or $x\to-\phi_\mu(\alpha,\beta;x)$ is absolutely monotonic. According to Hardy, Littlewood and Pólya theorem [24, Proposition 2.3.3] absolute monotonicity of $x\to\phi_\mu(\alpha,\beta;x)$ implies that this function is multiplicatively convex:

$$\phi_\mu(\alpha,\beta;x^\lambda y^{1-\lambda})\le\phi_\mu(\alpha,\beta;x)^\lambda\phi_\mu(\alpha,\beta;y)^{1-\lambda}$$

for $\lambda\in[0,1]$. Curiously, this inequality leads to interesting results even when applied to the simplest function $1+x^2$. We have

$$1+x^{2\lambda}y^{2(1-\lambda)}\le(1+x^2)^\lambda(1+y^2)^{1-\lambda}\ \text{ and }\ 1+x^{2(1-\lambda)}y^\lambda\le(1+x^2)^{1-\lambda}(1+y^2)^\lambda.$$

Multiplying these inequalities and simplifying we obtain

$$(x^\lambda y^{1-\lambda})^2+(x^{1-\lambda}y^\lambda)^2\le x^2+y^2,$$

which is equivalent to inequality

$$M_2(G_\lambda(x,y),G_{1-\lambda}(x,y))\le M_2(x,y),\quad x,y\ge 0,\quad 0\le\lambda\le 1,$$

where $M_2(a,b)=\sqrt{(a^2+b^2)/2}$ is quadratic mean, $G_\lambda(a,b)=a^\lambda b^{1-\lambda}$ is weighted geometric mean.

The paper is organized as follows: in section 2 we collect several lemmas repeatedly used in the proofs; section 3 comprises six theorems constituting the main content of the paper; in section 4 we present several applications and relate them to some known results.

2. Preliminaries. We will need several lemmas which we prove in this section.

Lemma 1 *Suppose $\{f(\mu;x)\}_{\mu\ge 0}$ and $\{g(\mu;x)\}_{\mu\ge 0}$ are (discrete, Wright) q-log-concave. Then $\{f(\mu;x)g(\mu;x)\}_{\mu\ge 0}$ is (discrete, Wright) q-log-concave.*

Remark. Lemma 1 holds, in particular, if $\mu \to g(\mu)$ is a log-concave function independent of x.
Proof. We have

$$
\begin{aligned}
f(\mu+\alpha;x)g(\mu+\alpha;x)f(\mu+\beta;x)g(\mu+\beta;x) - f(\mu;x)g(\mu;x)f(\mu+\alpha+\beta;x)g(\mu+\alpha+\beta;x) = \\
g(\mu+\alpha;x)g(\mu+\beta;x)(f(\mu+\alpha;x)f(\mu+\beta;x) - f(\mu;x)f(\mu+\alpha+\beta;x)) \\
+ f(\mu;x)f(\mu+\alpha+\beta;x)(g(\mu+\alpha;x)g(\mu+\beta;x) - g(\mu;x)g(\mu+\alpha+\beta;x)).
\end{aligned}
$$

This formula implies the claim of the lemma. $\square$

Lemma 2 *Let f be a nonnegative-valued function defined on $\mathbb{R}_+$. Suppose*

$$\phi_\mu(\alpha,\beta) := f(\mu+\alpha)f(\mu+\beta) - f(\mu)f(\mu+\beta+\alpha) \geq 0 \text{ for } \alpha = 1 \text{ and all } \mu,\beta \geq 0.$$

Then $\phi_\mu(\alpha,\beta) \geq 0$ for all $\alpha \in \mathbb{N}$ and $\mu,\beta \geq 0$, i.e. $\mu \to f(\mu)$ is discrete Wright log-concave on $\mathbb{R}_+$.

Proof. According to assumptions of the lemma written for the pairs $\{\mu,\beta\}$, $\{\mu+1,\beta\}$, $\{\mu,\beta+1\}$ and $\alpha = 1$ we have

$$f(\mu+1)f(\mu+\beta) \geq f(\mu)f(\mu+\beta+1), \tag{4}$$

$$f(\mu+2)f(\mu+\beta+1) \geq f(\mu+1)f(\mu+\beta+2), \tag{5}$$

$$f(\mu+1)f(\mu+\beta+1) \geq f(\mu)f(\mu+\beta+2). \tag{6}$$

A product of (4) and (5) reads

$$f(\mu+1)f(\mu+\beta+1)(f(\mu+\beta)f(\mu+2) - f(\mu)f(\mu+\beta+2)) \geq 0.$$

This implies either $f(\mu+\beta)f(\mu+2) \geq f(\mu)f(\mu+\beta+2)$ which is our claim for $\alpha = 2$ or $f(\mu+1)f(\mu+\beta+1) = 0$ which implies $f(\mu)f(\mu+\beta+2) = 0$ according to (6), so that again $f(\mu+\beta)f(\mu+2) \geq f(\mu)f(\mu+\beta+2)$. Hence, we have demonstrated that $\phi_\mu(2,\beta) \geq 0$. In a similar fashion $\phi_\mu(\alpha,\beta) \geq 0$ holds for all $\alpha \in \mathbb{N}$ and $\mu,\beta \geq 0$. $\square$

In the above Lemma the function f may or may not be defined by the series (1) - we have not made any use of the special series structure in the proof. In the next lemma we deal with Wright q-log-concavity and the series definition becomes important.

Lemma 3 *Let the series f be defined by* (1) *and suppose*

$$\phi_\mu(1,\beta;x) := f(\mu+1;x)f(\mu+\beta;x) - f(\mu;x)f(\mu+\beta+1;x)$$

has nonnegative coefficients at powers of x for and all $\mu,\beta \geq 0$. Then $\phi_\mu(\alpha,\beta;x)$ has nonnegative coefficients at powers of x for all $\alpha \in \mathbb{N}$, $\beta \geq \alpha - 1$ and $\mu \geq 0$.

Proof. Define

$$\psi_{\nu,x}(a,b) := f(\nu;x)^2 - f(\nu-a;x)f(\nu+b;x).$$

By assumptions of the lemma the difference

$$
\begin{aligned}
\psi_{\nu,x}(a,b) - \psi_{\nu,x}(a-1,b-1) &= f(\nu-a+1;x)f(\nu+b-1;x) - f(\nu-a;x)f(\nu+b;x) \\
&\stackrel{\mu:=\nu-a}{=} f(\mu+1;x)f(\mu+a+b-1;x) - f(\mu;x)f(\mu+a+b;x)
\end{aligned}
$$

has nonnegative power series coefficients when $\nu \geq a$, $a+b-1 \geq 0$. Further, for a positive integer k

$$f(\mu+k;x)f(\mu+\beta;x)-f(\mu;x)f(\mu+\beta+k;x) = f(\nu-a+k;x)f(\nu+b-k;x)-f(\nu-a;x)f(\nu+b;x)$$
$$= \psi_{\nu,x}(a,b) - \psi_{\nu,x}(a-k,b-k) = (\psi_{\nu,x}(a,b) - \psi_{\nu,x}(a-1,b-1))+$$
$$(\psi_{\nu,x}(a-1,b-1) - \psi_{\nu,x}(a-2,b-2)) + \cdots + (\psi_{\nu,x}(a-k+1,b-k+1) - \psi_{\nu,x}(a-k,b-k)),$$

where $\mu = \nu - a$, $\mu + \beta = \nu + b - k$. We must require $\nu \geq a$, $a+b-1 \geq 0$ for the first parentheses to have nonnegative power series coefficients, $\nu \geq a-1$, $a+b-3 \geq 0$ for the second parentheses to have nonnegative power series coefficients, and so on up to $\nu \geq a-k+1$, $a+b-2k+1 \geq 0$. These inequalities reduce to $\mu \geq 0$ and $\beta \geq k-1$. $\square$

The next lemma is implied by a much stronger result of Alzer [1, Theorem 10].

Lemma 4 *Suppose* $0 \leq \min(\alpha_1, \alpha_2) \leq \min(\beta_1, \beta_2)$ *and* $\alpha_1 + \alpha_2 \leq \beta_1 + \beta_2$. *Then the function*

$$x \to \frac{\Gamma(x+\alpha_1)\Gamma(x+\alpha_2)}{\Gamma(x+\beta_1)\Gamma(x+\beta_2)}$$

is strictly monotone decreasing on $(0, \infty)$*, except when the sets* $\{\alpha_1, \alpha_2\}$ *and* $\{\beta_1, \beta_2\}$ *are equal.*

Next, we formulate an elementary inequality we will repeatedly use below.

Lemma 5 *Suppose* $u, v, r, s > 0$*,* $u = \max(u, v, r, s)$ *and* $uv > rs$*. Then* $u + v > r + s$*.*

Lemma 5 is a particular case of a much more general result on logarithmic majorization - see [19, 2.A.b]. See also [14, Lemma 1] for a direct proof.

In the next lemma we say that a sequence has no more than one change of sign if it has the pattern $(- - \cdots - -00 \cdots 00 + + \cdots + +)$, where zeros and minus signs may be missing.

Lemma 6 *Suppose* $\{f_k\}_{k=0}^n$ *has no internal zeros and* $f_k^2 \geq f_{k-1}f_{k+1}$*,* $k = 1, 2, \ldots, n-1$*. If the real sequence* $A_0, A_1, \ldots, A_{[n/2]}$ *satisfying* $A_{[n/2]} > 0$ *and* $\sum_{0 \leq k \leq n/2} A_k \geq 0$ *has no more than one change of sign, then*

$$\sum_{0 \leq k \leq n/2} f_k f_{n-k} A_k \geq 0. \tag{7}$$

Equality is only attained if $f_k = f_0 \alpha^k$*,* $\alpha > 0$*, and* $\sum_{0 \leq k \leq n/2} A_k = 0$*.*

A proof of this lemma is found in [14, Lemma 2].

The generalized hypergeometric function is defined by the series

$${}_pF_q\left(\begin{matrix} A \\ B \end{matrix} \middle| z\right) = {}_pF_q\left(A; B; z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} z^n, \tag{8}$$

where we write $A = (a_1, a_2, \ldots, a_p)$, $B = (b_1, b_2, \ldots, b_q)$ for brevity and $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$, $n \geq 1$, denotes the rising factorial. The series (8) converges in the entire complex plane if $p \leq q$ and in the unit disk if $p = q+1$. In the latter case its sum can be extended analytically to the whole complex plane cut along the ray $[1, \infty)$ [2, Chapter 2]. The series (8) is a particular case of (3) because $(a)_k = \Gamma(a+k)/\Gamma(a)$.

The next identity for the Kummer function ${}_1F_1$ is believed to be new and may be of independent interest.

Lemma 7 *The Kummer function ${}_1F_1$ satisfies the following identity*:

$$\begin{aligned}
&{}_1F_1(a+\mu;c+\mu;x){}_1F_1(a+1;c+1;x) - {}_1F_1(a+\mu+1;c+\mu+1;x){}_1F_1(a;c;x) \\
&= \frac{(c-a)x}{c(c+1)(c+\mu)(c+\mu+1)}\left[(c+\mu)(c+\mu+1){}_1F_1(a+1;c+2;x){}_1F_1(a+\mu+1;c+\mu+1;x)\right. \\
&\qquad \left. -c(c+1){}_1F_1(a+1;c+1;x){}_1F_1(a+\mu+1;c+\mu+2;x)\right]. \quad (9)
\end{aligned}$$

Proof. Apply the easily verifiable contiguous relations

$${}_1F_1(a;c;x) = {}_1F_1(a;c+1;x) + \frac{ax}{c(c+1)}{}_1F_1(a+1;c+2;x),$$

$${}_1F_1(a+\mu;c+\mu;x) = {}_1F_1(a+\mu+1;c+\mu+1;x) - \frac{(c-a)x}{(c+\mu)(c+\mu+1)}{}_1F_1(a+\mu+1;c+\mu+2;x)$$

and

$${}_1F_1(a+1;c+1;x) = {}_1F_1(a;c+1;x) + \frac{x}{c+1}{}_1F_1(a+1;c+2;x)$$

to the corresponding functions on the left hand side of (9). Expanding and collecting similar terms we can then rewrite the left-hand side of (9) as

$$\begin{aligned}
&\frac{(c-a)x}{c(c+1)(c+\mu)(c+\mu+1)}\left[(c+\mu)(c+\mu+1){}_1F_1(a+1;c+2;x){}_1F_1(a+\mu+1;c+\mu+1;x)\right. \\
&\qquad \left. -c((c+1){}_1F_1(a;c+1;x) + x{}_1F_1(a+1;c+2;x)){}_1F_1(a+\mu+1;c+\mu+2;x)\right].
\end{aligned}$$

Finally, applying here the contiguous relation

$$(c+1){}_1F_1(a;c+1;x) + x{}_1F_1(a+1;c+2;x) = (c+1){}_1F_1(a+1;c+1;x),$$

yields the right hand-side of (9). $\Box$

The next lemma has been proved using some ideas borrowed from [7].

Lemma 8 *The inequality*

$$\sum_{k=0}^{m} \frac{(a)_k(a+\mu)_{m-k}}{(b)_k(b+\mu)_{m-k}}\binom{m}{k}(m-2k+\mu) \geq 0, \tag{10}$$

holds for each integer $m \geq 1$ and all $\mu \geq 0$ if $b \geq a \geq 0$ or $a \geq b \geq 1$.

Proof. Denote

$$u_k = \frac{(a)_k(a+\mu)_{m-k}}{(b)_k(b+\mu)_{m-k}}.$$

If $a = b$ or $a = 0$ the claim is obvious. Suppose first that $b > a > 0$. Then $x \to (a+x)/(b+x)$ increasing so that for $k < m-k$

$$u_k > u_{m-k}, \quad \text{since} \quad \frac{(a+\mu+k)\cdots(a+\mu+m-k-1)}{(b+\mu+k)\cdots(b+\mu+m-k-1)} > \frac{(a+k)\cdots(a+m-k-1)}{(b+k)\cdots(b+m-k-1)}.$$

It follows that

$$\binom{m}{k}u_k(m-2k+\mu) + \binom{m}{m-k}u_{m-k}(2k-m+\mu) = \binom{m}{k}[(m-2k)(u_k-u_{m-k}) + \mu(u_k+u_{m-k})] > 0$$

for each $k \le m-k$ which proves the lemma for all $b \ge a \ge 0$. If $a > b \ge 1$ things become more complicated. In this case we will apply Abel's lemma (summation by parts) in the form [7]

$$\sum_{k=0}^{m}(\alpha_{k+1}-\alpha_k)\beta_k = \sum_{k=0}^{m}\alpha_{k+1}(\beta_k-\beta_{k+1}) + \alpha_{m+1}\beta_{m+1} - \alpha_0\beta_0.$$

Gosper's algorithm [9], [27, Chapter 5] produces the following antidifference which is easy to verify directly:

$$u_k(m-2k+\mu) = \alpha_{k+1}-\alpha_k, \text{ where } \alpha_k = \frac{(b-1)(b-1+\mu)(a)_k(a+\mu)_{m+1-k}}{(a-b+1)(b-1)_k(b-1+\mu)_{m+1-k}}, \; k=0,1,\ldots,m+1.$$

Next, setting $\beta_k = \binom{m}{k}$ we immediately obtain

$$\beta_k - \beta_{k+1} = \binom{m}{k} - \binom{m}{k+1} = \frac{2k+1-m}{m+1}\binom{m+1}{k+1}.$$

Hence, an application of Abel's lemma yields (we use the fact that $\beta_{-1} = \beta_{m+1} = 0$):

$$\begin{aligned}
&\sum_{k=0}^{m}\binom{m}{k}u_k(m-2k+\mu) = \sum_{k=0}^{m}\alpha_{k+1}(\beta_k-\beta_{k+1}) - \alpha_0\beta_0 \\
&= \frac{(b-1)(b-1+\mu)}{(a-b+1)}\left\{\sum_{k=0}^{m}\frac{(a)_{k+1}(a+\mu)_{m-k}}{(b-1)_{k+1}(b-1+\mu)_{m-k}}\frac{2k+1-m}{m+1}\binom{m+1}{k+1} - \frac{(a+\mu)_{m+1}}{(b-1+\mu)_{m+1}}\right\} \\
&\qquad = \frac{(b-1)(b-1+\mu)}{(a-b+1)}\left\{\sum_{k=-1}^{m}\frac{(a)_{k+1}(a+\mu)_{m-k}}{(b-1)_{k+1}(b-1+\mu)_{m-k}}\frac{2k+1-m}{m+1}\binom{m+1}{k+1}\right\} \\
&\qquad\qquad = \frac{(b-1)(b-1+\mu)}{(a-b+1)n}\left\{\sum_{j=0}^{n}\frac{(a)_j(a+\mu)_{n-j}}{(b-1)_j(b-1+\mu)_{n-j}}\binom{n}{j}(2j-n)\right\},
\end{aligned}$$

where $j = k+1$, $n = m+1$. If $a > b \ge 1$ and $0 \le j < n-j$ we have

$$\frac{(a)_j(a+\mu)_{n-j}}{(b-1)_j(b-1+\mu)_{n-j}} < \frac{(a)_{n-j}(a+\mu)_j}{(b-1)_{n-j}(b-1+\mu)_j}$$

$$\Leftrightarrow \frac{(a+\mu+j)\cdots(a+\mu+n-j-1)}{(b-1+\mu+j)\cdots(b+\mu+n-j-2)} < \frac{(a+j)\cdots(a+n-j-1)}{(b-1+j)\cdots(b+n-j-2)}$$

since $x \to (a+x)/(b+x)$ is decreasing. Hence,

$$\begin{aligned}
&\sum_{j=0}^{n}\frac{(a)_j(a+\mu)_{n-j}}{(b-1)_j(b-1+\mu)_{n-j}}\binom{n}{j}(2j-n) \\
&\qquad = \sum_{0\le j<n/2}\left(\frac{(a)_{n-j}(a+\mu)_j}{(b-1)_{n-j}(b-1+\mu)_j} - \frac{(a)_j(a+\mu)_{n-j}}{(b-1)_j(b-1+\mu)_{n-j}}\right)\binom{n}{j}(n-2j) > 0. \; \square
\end{aligned}$$

Remark. With more effort one can show that (10) remains valid if $a \ge b \ge 1/2$, but we will not use this fact in the present paper.

3. Main results. In this section we prove six general theorems for series in ratios of rising factorials and gamma functions. The power series expansions in this section are understood as formal, so that no questions of convergence are discussed. In applications the radii of convergence will usually be apparent. The results of this section are exemplified by concrete special functions in the subsequent section. The first two theorems deal with the class of series defined by

$$f_{a,c}(\mu;x) := \sum_{n=0}^{\infty} f_n \frac{(a+\mu)_n}{(c+\mu)_n} \frac{x^n}{n!}. \tag{11}$$

Since

$$\begin{aligned} f_{a,c}(\mu+\alpha;x)f_{a,c}(\mu+\beta;x) - f_{a,c}(\mu;x)f_{a,c}(\mu+\alpha+\beta;x) \\ = f_{a+\mu,c+\mu}(\alpha;x)f_{a+\mu,c+\mu}(\beta;x) - f_{a+\mu,c+\mu}(0;x)f_{a+\mu,c+\mu}(\alpha+\beta;x), \end{aligned}$$

there is no loss of generality in considering the product difference (2) in the form

$$\phi_{a,c}(\mu,\nu;x) := f_{a,c}(\mu;x)f_{a,c}(\nu;x) - f_{a,c}(0;x)f_{a,c}(\mu+\nu;x) = \sum_{m=1}^{\infty} \phi_m x^m. \tag{12}$$

Logarithmic concavity or convexity of $\mu \mapsto f_{a,c}(\mu;x)$ depends on the interrelation between a and c.

Theorem 1 *Suppose $c \geq a > 0$ and $\{f_n\}_{n=0}^{\infty}$ is a nonnegative log-concave sequence without internal zeros. Then the function $\mu \mapsto f_{a,c}(\mu;x)$ is discrete Wright log-concave on $[0,\infty)$ for each fixed $x > 0$. Moreover, the family $\{f_{a,c}(\mu;x)\}_{\mu\geq 0}$ is discrete q-log-concave, i.e. the function $x \to \phi_{a,c}(\mu,\nu;x)$ has nonnegative power series coefficients for all $\nu \in \mathbb{N}$ and $\mu \geq \nu - 1$ so that $x \to \phi_{a,c}(\mu,\nu;x)$ is absolutely monotonic and multiplicatively convex on $(0,\infty)$.*

Remark. It easy to see from the proof of the theorem that $\phi_m > 0$ for all $m \geq 1$ if $f_n > 0$ for all $n \geq 0$, $c > a$ and $\mu > 0$.

Proof. If $c = a$ the claim is obvious. Suppose $c > a > 0$. According to Lemmas 2 and 3 it is sufficient to consider the case $\nu = 1$. For a fixed integer $m \geq 1$ we have by the Cauchy product and Gauss summation:

$$\phi_m = \sum_{k=0}^{m} f_k f_{m-k} \underbrace{\left(\frac{(a+1)_k(a+\mu)_{m-k}}{(c+1)_k(c+\mu)_{m-k}k!(m-k)!} - \frac{(a)_k(a+\mu+1)_{m-k}}{(c)_k(c+\mu+1)_{m-k}k!(m-k)!}\right)}_{N_k} = \sum_{k=0}^{[m/2]} f_k f_{m-k} M_k,$$

where $M_k = N_k + N_{m-k}$ for $k < m/2$ and $M_k = N_k$ for $k = m/2$. We aim to apply Lemma 6 to prove that $\phi_m \geq 0$. First, we need to show that

$$\sum_{k=0}^{[m/2]} M_k = \sum_{k=0}^{m} N_k \geq 0. \tag{13}$$

Since, clearly,

$$\sum_{m=1}^{\infty} x^m \sum_{k=0}^{m} N_k = {}_1F_1(a+\mu;c+\mu;x)\,{}_1F_1(a+1;c+1;x) - {}_1F_1(a+\mu+1;c+\mu+1;x)\,{}_1F_1(a;c;x),$$

we are in the position to apply formula (9) from Lemma 7 which, after equating power series coefficients on both sides, yields:

$$\sum_{k=0}^{m+1} N_k = \frac{(c-a)}{c(c+1)(c+\mu)(c+\mu+1)} \times$$
$$\sum_{k=0}^{m} \Bigg(\underbrace{\frac{(c+\mu)(c+\mu+1)(a+1)_k(a+\mu+1)_{m-k}}{(c+2)_k(c+\mu+1)_{m-k}k!(m-k)!}}_{u_k} - \underbrace{\frac{c(c+1)(a+1)_k(a+\mu+1)_{m-k}}{(c+1)_k(c+\mu+2)_{m-k}k!(m-k)!}}_{r_k} \Bigg)$$
$$= \sum_{k=0}^{[m/2]}{}' (u_k + u_{k-m} - r_k - r_{k-m}).$$

Here the prime at the summation sign means that the term with $k = m/2$ (which only happens for even m) has multiplier $1/2$. This last term is positive since ($l = m/2$)

$$u_l > r_l \;\Leftrightarrow\; \frac{(c+\mu)(c+\mu+1)}{(c+2)_l(c+\mu+1)_l} > \frac{c(c+1)}{(c+1)_l(c+\mu+2)_l} \;\Leftrightarrow\; \frac{(c+\mu)(c+\mu+l+1)}{c(c+l+1)} > 1.$$

We claim that all other terms in the rightmost sum above are also positive by Lemma 5 with $u = u_k$, $v = u_{m-k}$, $r = r_k$, $s = r_{m-k}$. To verify the assumptions of Lemma 5 we need to show that $u_k > u_{k-m}$, $u_k > r_k$, $u_k > r_{m-k}$ and $u_k u_{k-m} > r_k r_{k-m}$. We have

$$u_k > u_{k-m} \;\Leftrightarrow\; \left.\frac{\Gamma(a+1+x)\Gamma(c+\mu+1+x)}{\Gamma(c+2+x)\Gamma(a+\mu+1+x)}\right|_{x=k} > \left.\frac{\Gamma(a+1+x)\Gamma(c+\mu+1+x)}{\Gamma(c+2+x)\Gamma(a+\mu+1+x)}\right|_{x=m-k},$$

since the gamma quotient is decreasing by Lemma 4 and $k < m-k$ by assumption. Next,

$$u_k > r_k \;\Leftrightarrow\; (c+\mu)(c+\mu+1+m-k) > c(c+1+k),$$

which is true by $\mu > 0$ and $k < m-k$. The inequality $u_k > r_{m-k}$ reduces to

$$\left.\frac{(c+\mu)(c+\mu+k+1)\Gamma(a+1+x)\Gamma(c+\mu+1+x)}{c(c+k+1)\Gamma(a+\mu+1+x)\Gamma(c+1+x)}\right|_{x=k} > \left.\frac{\Gamma(a+1+x)\Gamma(c+\mu+1+x)}{\Gamma(a+\mu+1+x)\Gamma(c+1+x)}\right|_{x=m-k},$$

which is true because the gamma quotient is decreasing by Lemma 4 while $(c+\mu)(c+\mu+k+1)/[c(c+k+1)] \geq 1$. Finally,

$$u_k u_{k-m} > r_k r_{k-m} \;\Leftrightarrow\; (c+\mu)^2(c+\mu+k+1)(c+\mu+m-k+1) > c^2(c+k+1)(c+m-k+1),$$

which proves inequality (13).

Next, we need to demonstrate that the sequence $\{M_k\}_{k=0}^{[m/2]}$ changes sign not more than once. To this end introduce the following notation

$$\widetilde{M}_k = k!(m-k)!M_k = \begin{cases} \underbrace{\dfrac{(a+1)_k(a+\mu)_{m-k}}{(c+1)_k(c+\mu)_{m-k}}}_{=\widetilde{u}_k} + \underbrace{\dfrac{(a+1)_{m-k}(a+\mu)_k}{(c+1)_{m-k}(c+\mu)_k}}_{=\widetilde{u}_{m-k}} & \\ -\underbrace{\dfrac{(a)_k(a+\mu+1)_{m-k}}{(c)_k(c+\mu+1)_{m-k}}}_{=\widetilde{r}_k} - \underbrace{\dfrac{(a)_{m-k}(a+\mu+1)_k}{(c)_{m-k}(c+\mu+1)_k}}_{=\widetilde{r}_{m-k}}, & k < m/2 \\ \dfrac{(a+1)_{m/2}(a+\mu)_{m/2}}{(c+1)_{m/2}(c+\mu)_{m/2}} - \dfrac{(a)_{m/2}(a+\mu+1)_{m/2}}{(c)_{m/2}(c+\mu+1)_{m/2}}, & k = m/2. \end{cases}$$

Suppose that $\widetilde{M}_k < 0$ for some $0 < k < m/2$ then we will show that $\widetilde{M}_{k-1} < 0$. Indeed, $\widetilde{M}_{k-1}$ can be written in the following form

$$\widetilde{M}_{k-1} = \underbrace{\frac{(c+k)(a+\mu+m-k)}{(a+k)(c+\mu+m-k)}}_{=I_1}\widetilde{u}_k + \underbrace{\frac{(a+1+m-k)(c+\mu+k-1)}{(c+1+m-k)(a+\mu+k-1)}}_{=I_2}\widetilde{u}_{m-k} -$$

$$-\underbrace{\frac{(c+k-1)(a+\mu+1+m-k)}{(a+k-1)(c+\mu+1+m-k)}}_{=I_3}\widetilde{r}_k - \underbrace{\frac{(a+m-k)(c+\mu+k)}{(c+m-k)(a+\mu+k)}}_{=I_4}\widetilde{r}_{m-k} =$$

$$= I_1(\widetilde{u}_k+\widetilde{u}_{m-k}-\widetilde{r}_k-\widetilde{r}_{m-k})+(I_1-I_3)(\widetilde{r}_k-\widetilde{u}_{m-k})+(I_2-I_3)(\widetilde{u}_{m-k}-\widetilde{r}_{m-k})+(I_1+I_2-I_3-I_4)\widetilde{r}_{m-k}.$$

The first term is negative since $\widetilde{M}_k < 0$. We will show that all other terms are also negative. The function $x \mapsto \dfrac{\beta+x}{\alpha+x}$, $\beta > \alpha$, is strictly decreasing on $(0,\infty)$ which leads to the following inequalities

$$I_1 < I_3 \Leftrightarrow \frac{(c+k)(a+\mu+m-k)}{(a+k)(c+\mu+m-k)} < \frac{(c+k-1)(a+\mu+1+m-k)}{(a+k-1)(c+\mu+1+m-k)},$$

$$I_2 < I_3 \Leftrightarrow \frac{(a+1+m-k)(c+\mu+k-1)}{(c+1+m-k)(a+\mu+k-1)} < \frac{(c+k-1)(a+\mu+1+m-k)}{(a+k-1)(c+\mu+1+m-k)},$$

$$I_4 < I_2 \Leftrightarrow \frac{(a+m-k)(c+\mu+k)}{(c+m-k)(a+\mu+k)} < \frac{(a+1+m-k)(c+\mu+k-1)}{(c+1+m-k)(a+\mu+k-1)},$$

valid for $0 < k < m/2$ and $\mu > 0$. Hence, $I_3 = \max(I_1, I_2, I_3, I_4)$. Further, $I_3 I_4 > I_1 I_2$ is equivalent to

$$H_1(\mu) := \frac{(a+\mu+k-1)(c+\mu+k)(c+\mu+m-k)(a+\mu+1+m-k)}{(c+\mu+k-1)(a+\mu+k)(a+\mu+m-k)(a+\mu+1+m-k)} > H_1(0).$$

We will show that $H_1(\mu)$ is increasing on $(0,\infty)$. Indeed, by straightforward calculation

$$\frac{d}{d\mu}\log(H_1(\mu)) = (c-a)(H_2(z_2+1) - H_2(z_2) - (H_2(z_1+1) - H_2(z_1))),$$

where $0 \le z_1 = k-1 < z_2 = m-k$ and

$$H_2(z) = \frac{1}{(a+\mu+z)(c+\mu+z)}$$

is convex on $[0,\infty)$ implying $H_2(z_2+1)-H_2(z_2) > H_2(z_1+1)-H_2(z_1)$. Thus we have proved that $I_3 I_4 > I_1 I_2$ so that by Lemma 5 we get $I_1 + I_2 - I_3 - I_4 < 0$. To demonstrate that $\widetilde{M}_{k-1} < 0$ it remains to show that $\widetilde{u}_{m-k} > \widetilde{r}_{m-k}$ and $\widetilde{r}_k > \widetilde{u}_{m-k}$. We have

$$\widetilde{u}_{m-k} > \widetilde{r}_{m-k} \Leftrightarrow \frac{(a+1)_{m-k}(a+\mu)_k}{(c+1)_{m-k}(c+\mu)_k} > \frac{(a)_{m-k}(a+\mu+1)_k}{(c)_{m-k}(c+\mu+1)_k} \Leftrightarrow \frac{(a+m-k)c}{a(c+m-k)} > \frac{(a+\mu+k)(c+\mu)}{(a+\mu)(c+\mu+k)}.$$

Since $\mu \mapsto \dfrac{(a+\mu+k)(c+\mu)}{(a+\mu)(c+\mu+k)}$ is strictly decreasing on $[0,\infty)$

$$\frac{(a+\mu+k)(c+\mu)}{(a+\mu)(c+\mu+k)} < \frac{(a+k)c}{a(c+k)} < \frac{(a+m-k)c}{a(c+m-k)},$$

where the rightmost inequality clearly holds for $0 < k < m/2$. Finally, in order to show that $\widetilde{r}_k > \widetilde{u}_{m-k}$ it suffices to prove that $\widetilde{u}_k > \widetilde{r}_{m-k}$. Indeed, $\widetilde{u}_{m-k} \ge \widetilde{r}_k$ and the preceding inequality imply that $\widetilde{M}_k > 0$ contradicting our hypothesis. The validity of $\widetilde{u}_k > \widetilde{r}_{m-k}$ follows from

$$\widetilde{u}_k > \widetilde{r}_{m-k} \;\Leftrightarrow\; \frac{(a+1)_k(a+\mu)_{m-k}}{(c+1)_k(c+\mu)_{m-k}} > \frac{(a)_{m-k}(a+\mu+1)_k}{(c)_{m-k}(c+\mu+1)_k} \;\Leftrightarrow$$

$$\Leftrightarrow \ \frac{c\Gamma(a+1+k)\Gamma(a+\mu+m-k)}{a\Gamma(a+m-k)\Gamma(a+\mu+1+k)} > \frac{(c+\mu)\Gamma(c+1+k)\Gamma(c+\mu+m-k)}{(a+\mu)\Gamma(c+m-k)\Gamma(c+\mu+1+k)}.$$

It is easy to see that conditions of Lemma 4 are satisfied for the gamma ratio for all $0 \le k < m/2$ and $\mu > 0$, while clearly $c/a > (c+\mu)/(a+\mu)$. $\square$

Corollary 1 *Suppose $c > a > 0$ and the series in (11) converges for all $x \ge 0$. Then for all $\nu \in \mathbb{N}$ and $\mu \ge \nu - 1$ the function $y \to \phi_{a,c}(\mu,\nu;1/y)$ is completely monotonic and log-convex on $[0,\infty)$ and there exists a nonnegative measure τ supported on $[0,\infty)$ such that*

$$\phi_{a,c}(\mu,\nu;x) = \int\limits_{[0,\infty)} e^{-t/x} d\tau(t).$$

Proof. According to [20, Theorem 3] a convergent series of completely monotonic functions with nonnegative coefficients is again completely monotonic. This implies that $y \to \phi_{a,c}(\mu,\nu;1/y)$ is completely monotonic, so that the above integral representation follows by Bernstein's theorem [29, Theorem 1.4]. Log-convexity follows from complete monotonicity according to [24, Exersice 2.1(6)]. $\square$

Corollary 2 *Under hypotheses and notation of Theorem 1 and for all $\nu \in \mathbb{N}$, $\mu \ge \nu - 1$ and $x \ge 0$*

$$f_{a,c}(\mu;x)f_{a,c}(\nu;x) - f_{a,c}(0;x)f_{a,c}(\mu+\nu;x) \ge \frac{f_0 f_1 x\mu\nu(c-a)(2c+\mu+\nu)}{c(c+\mu)(c+\nu)(c+\mu+\nu)}$$

with equality only at $x = 0$ if $c - a, \mu, \nu \ne 0$.

Proof. Indeed, the claimed inequality is just $\phi_{a,c}(\mu,\nu;x) \ge \phi_1 x$ which is true by Theorem 1. $\square$

There is virtually no doubt that the discrete q-log-concavity demonstrated in Theorem 1 results from our method of proof so that the adjective "discrete" is redundant. In other words, we propose the following conjecture.

Conjecture 1 *The family $\{f(\mu;x)\}_{\mu \ge 0}$ is Wright q-log-concave for all $c \ge a > 0$.*

Next theorem handles the case $a \ge c > 0$. As it turns out frequently the log-convexity case is simpler.

Theorem 2 *Suppose $a \ge c > 0$, $\{f_n\}_{n=0}^{\infty}$ is any nonnegative sequence and the functions $f_{a,c}(\mu;x)$ and $\phi_{a,c}(\mu,\nu;x)$ are defined by (11) and (12), respectively. Then the function $\mu \mapsto f_{a,c}(\mu;x)$ is strictly log-convex on $[0,\infty)$ for each fixed $x > 0$. Moreover, the family $\{f_{a,c}(\mu;x)\}_{\mu \ge 0}$ is Wright q-log-convex, i.e. the function $x \to \phi_{a,c}(\mu,\nu;x)$ has non-positive power series coefficients so that $x \to -\phi_{a,c}(\mu,\nu;x)$ is absolutely monotonic and multiplicatively convex on $(0,\infty)$.*

Proof. If $a = c$ the claim is obvious. Suppose $a > c > 0$. Combining the Cauchy product with the Gauss summation as in the proof of Theorem 1 the problem reduces to the inequality

$$-\phi_m = \sum_{k=0}^{[m/2]} \frac{f_k f_{m-k}}{k!(m-k)!} M_k > 0, \tag{14}$$

where

$$M_k = \underbrace{\frac{(a)_k(a+\mu+\nu)_{m-k}}{(c)_k(c+\mu+\nu)_{m-k}}}_{=v} + \underbrace{\frac{(a)_{m-k}(a+\mu+\nu)_k}{(c)_{m-k}(c+\mu+\nu)_k}}_{=u} - \underbrace{\frac{(a+\mu)_k(a+\nu)_{m-k}}{(c+\mu)_k(c+\nu)_{m-k}}}_{=r} - \underbrace{\frac{(a+\mu)_{m-k}(a+\nu)_k}{(c+\mu)_{m-k}(c+\nu)_k}}_{=s}$$

for $k < m/2$ and

$$M_k = \frac{(a)_k(a+\mu+\nu)_{m-k}}{(c)_k(c+\mu+\nu)_{m-k}} - \frac{(a+\mu)_k(a+\nu)_{m-k}}{(c+\mu)_k(c+\nu)_{m-k}} \quad \text{for } k = m/2$$

(this term is missing for odd values of m). We will show that $M_k > 0$ for each $k = 1, 2, \dots, m/2$. We will need the following fact [17, Lemma 2 and Remark 7]: the function

$$x \mapsto \frac{(x+\alpha_1)(x+\alpha_2)}{(x+\beta_1)(x+\beta_2)}, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0,$$

is increasing on $[0, \infty)$ iff

$$\frac{\beta_1\beta_2}{\alpha_1\alpha_2} \geq \frac{\beta_1+\beta_2}{\alpha_1+\alpha_2} \geq 1.$$

It follows that this function is bounded by 1 for all $x \geq 0$ since its value at infinity is 1. If any of the inequalities above is strict the function is strictly increasing. This implies that $M_k > 0$ if $k = m/2$ for

$$\frac{(a)_k(a+\mu+\nu)_k}{(a+\mu)_k(a+\nu)_k} > \frac{(c)_k(c+\mu+\nu)_k}{(c+\mu)_k(c+\nu)_k}.$$

Indeed, both sides of this inequality represent a product of the terms

$$\frac{(x+i)(x+\mu+\nu+i)}{(x+\mu+i)(x+\nu+i)}$$

Since $(\mu+i)(\nu+i) > i(\mu+\nu+i)$ for each nonnegative integer i, this function is increasing and its value at $x = a$ is greater then its value at $x = c < a$.

Further, we will show that $M_k > 0$ for $0 \leq k < m/2$ using Lemma 5. We have

$$u > v \;\Leftrightarrow\; \frac{\Gamma(c+k)\Gamma(a+\mu+\nu+k)}{\Gamma(a+k)\Gamma(c+\mu+\nu+k)} > \frac{\Gamma(c+m-k)\Gamma(a+\mu+\nu+m-k)}{\Gamma(a+m-k)\Gamma(c+\mu+\nu+m-k)}$$

by Lemma 4 for $a > c > 0$ and $k < m-k$;

$$u > s \;\Leftrightarrow\; \frac{(c+\nu)_k(a+\mu+\nu)_k}{(a+\nu)_k(c+\mu+\nu)_k} > \frac{(c)_{m-k}(a+\mu)_{m-k}}{(a)_{m-k}(c+\mu)_{m-k}}$$

because

$$\frac{(c+\nu+i)(a+\mu+\nu+i)}{(a+\nu+i)(c+\mu+\nu+i)} > \frac{(c+i)(a+\mu+i)}{(a+i)(c+\mu+i)} \text{ for } i = 0, 1, \dots, k-1,$$

and $0 < (c+i)(a+\mu+i)/[(a+i)(c+\mu+i)] < 1$ for $i = k, \dots, m-k-1$ by the fact above; Next, $u > r$ by exactly the same argument with μ and ν interchanged; finally,

$$uv > rs \;\Leftrightarrow\; \frac{(a)_k(a+\nu+\mu)_k}{(a+\nu)_k(a+\mu)_k} \times \frac{(a)_{m-k}(a+\nu+\mu)_{m-k}}{(a+\nu)_{m-k}(a+\mu)_{m-k}} > \frac{(c)_k(c+\nu+\mu)_k}{(c+\nu)_k(c+\mu)_k} \times \frac{(c)_{m-k}(c+\nu+\mu)_{m-k}}{(c+\nu)_{m-k}(c+\mu)_{m-k}},$$

where the first multiplier on the left is bigger than the first multiplier on the right and the second multiplier on the left is bigger than the second multiplier on the right due to the increase of $x \to (x+i)(x+\nu+\mu+i)/[(a+\nu+i)(a+\mu+i)]$ for each $i \geq 0$. Hence, by Lemma 5 $M_k = u+v-r-s > 0$. $\square$

The following two corollaries are similar to Corollaries 1 and 2.

Corollary 3 *Under hypotheses and notation of Theorem 2 and for all $\mu,\nu,x\geq 0$*

$$f_{a,c}(0;x)f_{a,c}(\mu+\nu;x)-f_{a,c}(\mu;x)f_{a,c}(\nu;x)\geq \frac{f_0f_1x\mu\nu(a-c)(2c+\mu+\nu)}{c(c+\mu)(c+\nu)(c+\mu+\nu)}$$

with equality only at $x=0$ if $a-c,\mu,\nu\neq 0$.

Corollary 4 *Suppose $a>c>0$ and the series in* (11) *converges for all $x\geq 0$. Then for all $\nu,\mu>0$ the function $y\to-\phi_{a,c}(\mu,\nu;1/y)$ is completely monotonic and log-convex on $[0,\infty)$ and there exists a nonnegative measure τ supported on $[0,\infty)$ such that*

$$\phi_{a,c}(\mu,\nu;x)=-\int\limits_{[0,\infty)}e^{-t/x}d\tau(t).$$

The next two theorems deal with the class of series defined by

$$\mu\to g_{a,c}(\mu;x)=\sum_{n=0}^{\infty}g_n\frac{\Gamma(a+n+\mu)}{\Gamma(c+n+\mu)}\frac{x^n}{n!}\tag{15}$$

and their product differences

$$\psi_{a,c}(\mu,\nu;x)=g_{a,c}(\mu;x)g_{a,c}(\nu;x)-g_{a,c}(0;x)g_{a,c}(\mu+\nu;x)=\sum_{m=0}^{\infty}\psi_m x^m.\tag{16}$$

If we set $g_n=f_n$ we get

$$g_{a,c}(\mu;x)=\frac{\Gamma(a+\mu)}{\Gamma(c+\mu)}f_{a,c}(\mu;x)$$

where $f_{a,c}(\mu;x)$ is defined by (11). It is then tempting to derive the properties of $g_{a,c}(\mu;x)$ from Theorems 1 and 2 using Lemma 1. However, when $a>c$ the gamma ratio in front of $f_{a,c}(\mu;x)$ is log-concave while $\mu\to f_{a,c}(\mu;x)$ is log-convex, so that Lemma 1 cannot be applied. Similar situation holds for $c\geq a$.

Theorem 3 *Suppose $c\geq a>0$, $\{g_n\}_{n=0}^{\infty}$ is any nonnegative sequence. Then the function $\mu\to g_{a,c}(\mu;x)$ is Wright log-convex on $[0,\infty)$ for each fixed $x\geq 0$. Moreover, the family $\{g_{a,c}(\mu;x)\}_{\mu\geq 0}$ is Wright q-log-convex, i.e. the function $x\to\psi_{a,c}(\mu,\nu;x)$ has non-positive power series coefficients for all $\mu,\nu\geq 0$ so that $x\to-\psi_{a,c}(\mu,\nu;x)$ is absolutely monotonic and multiplicatively convex on $(0,\infty)$.*

Proof. Cauchy product and Gauss summation yield

$$-\psi_m=\sum_{k=0}^{m}\frac{g_kg_{m-k}}{k!(m-k)!}\left\{\frac{\Gamma(a+k)\Gamma(a+\nu+\mu+m-k)}{\Gamma(c+k)\Gamma(c+\nu+\mu+m-k)}-\frac{\Gamma(a+\nu+k)\Gamma(a+\mu+m-k)}{\Gamma(c+\nu+k)\Gamma(c+\mu+m-k)}\right\}$$
$$=\sum_{k=0}^{[m/2]}\frac{g_kg_{m-k}}{k!(m-k)!}M_k,$$

where

$$M_k=\underbrace{\frac{\Gamma(a+k)\Gamma(a+\nu+\mu+m-k)}{\Gamma(c+k)\Gamma(c+\nu+\mu+m-k)}}_{=u}+\underbrace{\frac{\Gamma(a+m-k)\Gamma(a+\nu+\mu+k)}{\Gamma(c+m-k)\Gamma(c+\nu+\mu+k)}}_{=v}$$
$$-\underbrace{\frac{\Gamma(a+\nu+k)\Gamma(a+\mu+m-k)}{\Gamma(c+\nu+k)\Gamma(c+\mu+m-k)}}_{=r}-\underbrace{\frac{\Gamma(a+\nu+m-k)\Gamma(a+\mu+k)}{\Gamma(c+\nu+m-k)\Gamma(c+\mu+k)}}_{=s}$$

for $k < m/2$, and

$$M_k = \frac{\Gamma(a+k)\Gamma(a+\nu+\mu+m-k)}{\Gamma(c+k)\Gamma(c+\nu+\mu+m-k)} - \frac{\Gamma(a+\nu+k)\Gamma(a+\mu+m-k)}{\Gamma(c+\nu+k)\Gamma(c+\mu+m-k)} \quad \text{for} \quad k = m/2$$

(this term is missing for odd values of m). We aim to demonstrate that $M_k > 0$ using Lemma 5. We have

$$u > v \;\Leftrightarrow\; \frac{\Gamma(a+k)\Gamma(c+\nu+\mu+k)}{\Gamma(c+k)\Gamma(a+\nu+\mu+k)} > \frac{\Gamma(a+m-k)(c+\nu+\mu+m-k)}{\Gamma(c+m-k)\Gamma(a+\nu+\mu+m-k)}.$$

In view of $k < m-k$, the last inequality holds by Lemma 4 with $\alpha_1 = c+\nu+\mu$, $\alpha_2 = a$, $\beta_1 = \max(c, a+\nu+\mu)$ $\beta_2 = \min(c, a+\nu+\mu)$, $x_1 = k$, $x_2 = m-k$. Next,

$$u > r \;\Leftrightarrow\; \frac{\Gamma(a+k)\Gamma(a+\nu+\mu+m-k)}{\Gamma(a+\nu+k)\Gamma(a+\mu+m-k)} > \frac{\Gamma(c+k)(c+\nu+\mu+m-k)}{\Gamma(c+\nu+k)(c+\mu+m-k)}$$

Setting $\alpha_1 = \nu+\mu+m-k$, $\alpha_2 = k$, $\beta_1 = \max(\nu+k, \mu+m-k)$, $\beta_2 = \min(\nu+k, \mu+m-k)$, $x_1 = a$ and $x_2 = c$ we get the last inequality by Lemma 4 again. In a similar fashion one can demonstrate that $u > s$. Finally, $uv > rs$ by multiplication of the following two inequalities

$$\frac{\Gamma(a+k)\Gamma(a+\nu+\mu+k)}{\Gamma(a+\nu+k)\Gamma(a+\mu+k)} > \frac{\Gamma(c+k)\Gamma(c+\nu+\mu+k)}{\Gamma(c+\nu+k)\Gamma(c+\mu+k)}$$

and

$$\frac{\Gamma(a+m-k)\Gamma(a+\nu+\mu+m-k)}{\Gamma(a+\mu+m-k)\Gamma(a+\nu+m-k)} > \frac{\Gamma(c+m-k)\Gamma(c+\nu+\mu+m-k)}{\Gamma(c+\mu+m-k)\Gamma(c+\nu+m-k)},$$

each of them holds by Lemma 4. $\square$

Again we have two corollaries similar to Corollaries 1 and 2.

Corollary 5 *Under hypotheses and notation of Theorem 3 and for all* $\mu, \nu, x \geq 0$

$$g_{a,c}(0;x)g_{a,c}(\mu+\nu;x) - g_{a,c}(\mu;x)g_{a,c}(\nu;x) \geq g_0^2 \left\{ \frac{\Gamma(a)\Gamma(a+\mu+\nu)}{\Gamma(c)\Gamma(c+\mu+\nu)} - \frac{\Gamma(a+\nu)\Gamma(a+\mu)}{\Gamma(c+\nu)\Gamma(c+\mu)} \right\}$$

with equality only at $x = 0$ *if* $c-a, \mu, \nu \neq 0$.

Corollary 6 *Suppose* $a > c > 0$ *and the series in* (15) *converges for all* $x \geq 0$. *Then for all* $\nu, \mu > 0$ *the function* $y \to -\psi_{a,c}(\mu, \nu; 1/y)$ *is completely monotonic and log-convex on* $[0, \infty)$ *and there exists a nonnegative measure* τ *supported on* $[0, \infty)$ *such that*

$$\psi_{a,c}(\mu, \nu; x) = -\int\limits_{[0,\infty)} e^{-t/x} d\tau(t).$$

Next, we can combine Theorem 1 and Theorem 3 to get

Corollary 7 *Under hypotheses and notation of Theorem 1*

$$\frac{(c+\mu)_\nu (a)_\nu}{(a+\mu)_\nu (c)_\nu} \leq \frac{f_{a,c}(0;x) f_{a,c}(\mu+\nu;x)}{f_{a,c}(\nu;x) f_{a,c}(\mu;x)} \leq 1$$

for all $\nu \in \mathbb{N}$, $\mu \geq 0$ *and* $x \geq 0$.

Proof. The estimate from above is a restatement of $\phi_{a,c}(\mu,\nu;x)\geq 0$ valid by Theorem 1. To demonstrate the estimate from below set in Theorem 3

$$g_{a,c}(\mu;x)=\frac{\Gamma(a+\mu)}{\Gamma(c+\mu)}f_{a,c}(\mu;x).$$

In view of $(a)_k=\Gamma(a+k)/\Gamma(a)$ the required inequality is a restatement of $\psi_{a,c}(\mu,\nu;x)\leq 0$. □

Further, combining Corollary 2 with Corollary 7 we obtain the following two-sided bounds for the Turánian:

$$\frac{2xf_0f_1\nu^2(c-a)}{c(c+\nu)(c+2\nu)}\leq f_{a,c}(\nu;x)^2-f_{a,c}(0;x)f_{a,c}(2\nu;x)\leq\frac{(a+\nu)_\nu(c)_\nu-(c+\nu)_\nu(a)_\nu}{(c)_\nu(a+\nu)_\nu}f_{a,c}(\nu;x)^2. \quad (17)$$

This holds for $\nu\in\mathbb{N}$, $c\geq a>0$, $x\geq 0$ and a log-concave sequence $\{f_n\}_{n\geq 0}$ without internal zeros. Indeed setting $\mu=\nu$ in Corollary 2 we get the lower bound, while setting $\mu=\nu$ in Corollary 7 multiplying throughout by $f_{a,c}(\nu;x)^2$ and subtracting the same expression we get the upper bound.

Remark. Theorems 2 and 3 are independent in the sense that

Theorem 4 *Suppose either* (a) $c+1\geq a\geq c>0$ *and* $\{g_n\}_{n=0}^\infty$ *is an arbitrary nonnegative sequence or* (b) $a>c+1>1$ *and* $\{g_n\}_{n=0}^\infty$ *is a nonnegative log-concave sequence without internal zeros. Then* $\mu\mapsto g_{a,c}(\mu;x)$ *is discrete Wright log-concave on* $[0,\infty)$ *for each fixed* $x>0$. *Moreover, the family* $\{g_{a,c}(\mu;x)\}_{\mu\geq 0}$ *is discrete q-log-concave, i.e. the function* $x\to\psi_{a,c}(\mu,\nu;x)$ *has nonnegative power series coefficients for all* $\nu\in\mathbb{N}$ *and* $\mu\geq\nu-1$ *so that* $x\to\psi_{a,c}(\mu,\nu;x)$ *is absolutely monotonic and multiplicatively convex on* $(0,\infty)$.

Proof. According to Lemmas 2 and 3 it is sufficient to consider the case $\nu=1$. For a fixed integer $m\geq 1$ we have by the Cauchy product and Gauss summation:

$$\begin{aligned}
\psi_m&=\sum_{k=0}^{m}\frac{g_kg_{m-k}}{k!(m-k)!}\left[\frac{\Gamma(a+1)(a+1)_k\Gamma(a+\mu)(a+\mu)_{m-k}}{\Gamma(c+1)(c+1)_k\Gamma(c+\mu)(c+\mu)_{m-k}}-\frac{\Gamma(a)(a)_k\Gamma(a+\mu+1)(a+\mu+1)_{m-k}}{\Gamma(c)(c)_k\Gamma(c+\mu+1)(c+\mu+1)_{m-k}}\right]\\
&=\frac{\Gamma(a)\Gamma(a+\mu)}{\Gamma(c)\Gamma(c+\mu)}\sum_{k=0}^{m}\frac{g_kg_{m-k}}{k!(m-k)!}\left[\frac{a(a+1)_k(a+\mu)_{m-k}}{c(c+1)_k(c+\mu)_{m-k}}-\frac{(a)_k(a+\mu)(a+\mu+1)_{m-k}}{(c)_k(c+\mu)(c+\mu+1)_{m-k}}\right]\\
&=\frac{\Gamma(a)\Gamma(a+\mu)}{\Gamma(c)\Gamma(c+\mu)}\sum_{k=0}^{m}\frac{g_kg_{m-k}}{k!(m-k)!}\frac{(a)_k(a+\mu)_{m-k}}{(c)_k(c+\mu)_{m-k}}\left[\frac{(a+k)}{(c+k)}-\frac{(a+\mu+m-k)}{(c+\mu+m-k)}\right]\\
&=\frac{\Gamma(a)\Gamma(a+\mu)}{\Gamma(c)\Gamma(c+\mu)}\sum_{k=0}^{m}\frac{g_kg_{m-k}}{k!(m-k)!}\frac{(a)_k(a+\mu)_{m-k}}{(c)_k(c+\mu)_{m-k}}\frac{(a-c)(m-2k+\mu)}{(c+k)(c+\mu+m-k)}\\
&=\frac{(a-c)\Gamma(a)\Gamma(a+\mu)}{\Gamma(c+1)\Gamma(c+\mu+1)m!}\sum_{k=0}^{m}\frac{g_kg_{m-k}(a)_k(a+\mu)_{m-k}}{(c+1)_k(c+1+\mu)_{m-k}}\binom{m}{k}(m-2k+\mu)\\
&=\frac{(a-c)\Gamma(a)\Gamma(a+\mu)}{\Gamma(c+1)\Gamma(c+\mu+1)m!}\sum_{k=0}^{[m/2]}g_kg_{m-k}\binom{m}{k}M_k,
\end{aligned}$$

where

$$M_k=\underbrace{\frac{(a)_k(a+\mu)_{m-k}}{(c+1)_k(c+1+\mu)_{m-k}}}_{=V_k}(m-2k+\mu)-\underbrace{\frac{(a)_{m-k}(a+\mu)_k}{(c+1)_{m-k}(c+1+\mu)_k}}_{=V_{m-k}}(m-2k-\mu).$$

for $k<m/2$ and

$$M_k=\frac{(a)_k(a+\mu)_{m-k}\mu}{(c+1)_k(c+1+\mu)_{m-k}}$$

for $k = m/2$. Lemma 8 shows that

$$\sum_{0 \le k \le m/2} \binom{m}{k} M_k > 0$$

for all $a > c > 0$. Moreover, the proof of the lemma for the case $c + 1 = b \ge a > 0$ implies that $M_k > 0$ for all $k = 0, 2, \ldots, [m/2]$. This proves the first part of the theorem. In order to prove the second part pertaining to $a > c + 1$ we will apply Lemma 6. Setting $A_k = \binom{m}{k} M_k$ it is left to demonstrate that that sequence $M_0, M_1, \ldots, M_{[m/2]}$ changes sign no more than once. Indeed for $k = m - k$ it is clear that $M_k > 0$. If $k < m - k$ then

$$V_k < V_{m-k} \ \Leftrightarrow \ \frac{(a+\mu+k)\cdots(a+\mu+m-k-1)}{(c+1+\mu+k)\cdots(c+1+\mu+m-k-)} < \frac{(a+k)\cdots(a+m-k-1)}{(c+1+k)\cdots(c+1+m-k-1)}$$

since $x \to (a+x)/(c+1+x)$ is decreasing. Assume that $M_k < 0$ for some k. We will demonstrate that $M_{k-1} < 0$. We have

$$M_{k-1} = V_k R(\mu)(m - 2k + \mu + 2) - V_{m-k} S(\mu)(m - 2k - \mu + 2),$$

where

$$R(\delta) = \frac{(a+\delta+m-k)(c+k)}{(c+1+\delta+m-k)(a+k-1)}, \quad S(\delta) = \frac{(c+\delta+k)(a+m-k)}{(a-1+\delta+k)(c+1+m-k)}.$$

It follows from $R(0) = S(0)$ and $V_k < V_{m-k}$ that

$$V_k R(0)(m-2k+\mu+2) - V_{m-k} S(0)(m-2k-\mu+2) = R(0)(M_k + 2(V_k - V_{m-k})) < 0.$$

Next, $R(\delta)$ is decreasing, while $S(\delta)$ is increasing on $(0, \infty)$ because $a > c+1$, and $m - 2k - \mu > 0$ because $M_k < 0$ so that

$$\begin{aligned} M_{k-1} &= V_k R(\mu)(m-2k+\mu) - V_{m-k} S(\mu)(m-2k-\mu) + 2(V_k R(\mu) - V_{m-k} S(\mu)) \\ &< R(0)(M_k + 2(V_k - V_{m-k})) < 0. \quad \square \end{aligned}$$

Again we have the following corollaries.

Corollary 8 *Suppose $\nu \in \mathbb{N}$ and $\mu \ge \nu - 1$. Under hypotheses of Theorem 4 the function $y \to \psi_{a,c}(\mu, \nu; 1/y)$ is completely monotonic and log-convex on $[0, \infty)$ and there exists a nonnegative measure τ supported on $[0, \infty)$ such that*

$$\psi_{a,c}(\mu, \nu; x) = \int\limits_{[0,\infty)} e^{-t/x} d\tau(t).$$

Corollary 9 *Under hypotheses and notation of Theorem 4 and for all $\nu \in \mathbb{N}$, $\mu \ge \nu - 1$, $x \ge 0$*

$$g_{a,c}(\mu; x) g_{a,c}(\nu; x) - g_{a,c}(0; x) g_{a,c}(\mu+\nu; x) \ge g_0^2 \left\{ \frac{\Gamma(a+\nu)\Gamma(a+\mu)}{\Gamma(c+\nu)\Gamma(c+\mu)} - \frac{\Gamma(a)\Gamma(a+\mu+\nu)}{\Gamma(c)\Gamma(c+\mu+\nu)} \right\}$$

with equality only at $x = 0$ if $a - c, \mu, \nu \ne 0$.

Corollary 10 *Under hypotheses and notation of Theorem 4*

$$\frac{(a+\mu)_\nu (c)_\nu}{(c+\mu)_\nu (a)_\nu} \le \frac{g_{a,c}(0; x) g_{a,c}(\mu+\nu; x)}{g_{a,c}(\nu; x) g_{a,c}(\mu; x)} \le 1$$

for all $\nu \in \mathbb{N}$, $\mu \ge 0$ and $x \ge 0$.

Combining corollaries 9 and 10 we obtain the following two-sided bounds for the Turánian:

$$g_0^2\frac{\Gamma(a)^2}{\Gamma(c)^2}\left[\frac{(a)_\nu^2}{(c)_\nu^2}-\frac{(a)_{2\nu}}{(c)_{2\nu}}\right]\leq g_{a,c}(\nu;x)^2-g_{a,c}(0;x)g_{a,c}(2\nu;x)\leq\frac{(c+\nu)_\nu(a)_\nu-(a+\nu)_\nu(c)_\nu}{(a)_\nu(c+\nu)_\nu}g_{a,c}(\nu;x)^2. \tag{18}$$

The bounds are valid for $\nu\in\mathbb{N}$, $a\geq c>0$ and assuming that $\{g_n\}_{n\geq 0}$ is a nonnegative sequence which is also log-concave and without internal zeros if $a>c+1$.

There is virtually no doubt that the discrete Wright log-concavity demonstrated in Theorem 4 results from our method of proof so that the adjective "discrete" is redundant. In other words, we propose the following conjecture.

Conjecture 2 *The family $\{g_{a,c}(\mu;x)\}_{\mu\geq 0}$ is Wright q-log-concave for all $a\geq c>0$.*

The next theorem deals with the class of series defined by

$$\mu\to h_{a,c}(\mu;x)=\sum_{n=0}^{\infty}h_n\frac{(a+\mu)_n}{\Gamma(c+\mu+n)}\frac{x^n}{n!} \tag{19}$$

and their product differences

$$\lambda_{a,c}(\mu,\nu;x)=h_{a,c}(\mu;x)h_{a,c}(\nu;x)-h_{a,c}(0;x)h_{a,c}(\nu+\mu;x)=\sum_{m=0}^{\infty}\lambda_m x^m. \tag{20}$$

Here we have discrete q-logarithmic concavity for all nonnegative values of a and c.

Theorem 5 *Suppose either* (a) $c+1\geq a\geq c>0$ *and $\{h_n\}_{n=0}^{\infty}$ is an arbitrary nonnegative sequence or* (b) $c>0$, $a\in(0,c)\cup(c+1,\infty)$ *and $\{h_n\}_{n=0}^{\infty}$ is a nonnegative log-concave sequence without internal zeros. Then $\mu\mapsto h_{a,c}(\mu;x)$ is discrete Wright log-concave on $[0,\infty)$ for each fixed $x>0$. Moreover, the family $\{h_{a,c}(\mu;x)\}_{\mu\geq 0}$ is discrete q-log-concave, i.e. $x\to\lambda_{a,c}(\mu,\nu;x)$ has nonnegative power series coefficients for $\nu\in\mathbb{N}$, $\mu\geq\nu-1$ so that $x\to\lambda_{a,c}(\mu,\nu;x)$ is absolutely monotonic and multiplicatively convex on $(0,\infty)$.*

Proof. Suppose $a\geq c>0$. We have

$$h_{a,c}(\mu;x)=\frac{1}{\Gamma(a+\mu)}g_{a,c}(\mu;x),$$

where $g_{a,c}(\mu;x)$ is defined in (15) and $g_n=h_n$. The claims of the theorem for $a\geq c>0$ then follow from Theorem 4 and Lemma 1.

If $c>a>0$ write

$$h_{a,c}(\mu;x)=\frac{1}{\Gamma(c+\mu)}f_{a,c}(\mu;x),$$

where $f_{a,c}(\mu;x)$ is defined in (11) and $f_n=h_n$. The claims of the theorem for $c>a>0$ then follow from Theorem 1 and Lemma 1. □

Corollary 11 *Under hypotheses of Theorem 5 and assuming the series in* (19) *converges for all $x\geq 0$ the function $y\to\lambda_{a,c}(\mu,\nu;1/y)$ is completely monotonic and log-convex on $[0,\infty)$ for all $\nu\in\mathbb{N}$ and $\mu\geq\nu-1$ so that there exists a nonnegative measure τ supported on $[0,\infty)$ such that*

$$\lambda_{a,c}(\mu,\nu;x)=\int\limits_{[0,\infty)}e^{-t/x}d\tau(t).$$

Corollary 12 *Under hypotheses of Theorem 5 and for all $\nu \in \mathbb{N}$, $\mu \geq \nu - 1$ and $x \geq 0$*

$$h_{a,c}(\mu;x)h_{a,c}(\nu;x) - h_{a,c}(0;x)h_{a,c}(\mu+\nu;x) \geq \frac{h_0^2[(c+\mu)_\nu - (c)_\nu]}{\Gamma(c+\nu)\Gamma(c+\mu+\nu)}$$

with equality only at $x = 0$ if $\mu, \nu \neq 0$.

Finally, we consider the the class of series defined by

$$\mu \to q_{a,c}(\mu;x) = \sum_{n=0}^{\infty} q_n \frac{\Gamma(a+\mu+n)}{(c+\mu)_n} \frac{x^n}{n!} \tag{21}$$

and their product differences

$$\rho_{a,c}(\mu,\nu;x) = q_{a,c}(\mu;x)q_{a,c}(\nu;x) - q_{a,c}(0;x)q_{a,c}(\nu+\mu;x) = \sum_{m=0}^{\infty} \rho_m x^m. \tag{22}$$

Here we have q-logarithmic convexity for all nonnegative values of a and c.

Theorem 6 *Suppose $a, c > 0$, $\{q_n\}_{n=0}^{\infty}$ is any nonnegative sequence and the functions $q_{a,c}(\mu;x)$ and $\rho_{a,c}(\mu,\nu;x)$ are defined by (21) and (22), respectively. Then the function $\mu \mapsto q_{a,c}(\mu;x)$ is strictly log-convex on $[0,\infty)$ for each fixed $x > 0$. Moreover, the family $\{q_{a,c}(\mu;x)\}_{\mu\geq 0}$ is Wright q-log-concave, i.e. the function $x \to \rho_{a,c}(\mu,\nu;x)$ has non-positive power series coefficients for all $\mu, \nu \geq 0$ so that the function $x \to -\rho_{a,c}(\mu,\nu;x)$ is absolutely monotonic and multiplicatively convex on $(0,\infty)$.*

Proof. Suppose $a \geq c > 0$. We have

$$q_{a,c}(\mu;x) = \Gamma(a+\mu)f_{a,c}(\mu;x),$$

where $f_{a,c}(\mu;x)$ is defined in (11) and $f_n = q_n$. The claims of the theorem for $a \geq c > 0$ then follow from Theorem 2 and Lemma 1.

If $c > a > 0$ write

$$q_{a,c}(\mu;x) = \Gamma(c+\mu)g_{a,c}(\mu;x),$$

where $g_{a,c}(\mu;x)$ is defined in (15) and $g_n = q_n$. The claims of the theorem for $c > a > 0$ then follow from Theorem 3 and Lemma 1. □

Corollary 13 *Under hypotheses of Theorem 6 and for all $x \geq 0$*

$$q_{a,c}(0;x)q_{a,c}(\mu+\nu;x) - q_{a,c}(\mu;x)q_{a,c}(\nu;x) \geq q_0^2 \{\Gamma(a)\Gamma(a+\mu+\nu) - \Gamma(a+\mu)\Gamma(a+\nu)\}$$

with equality only at $x = 0$ if $\mu, \nu \neq 0$.

Corollary 14 *Under hypotheses of Theorem 6 and assuming the series in (21) converges for all $x \geq 0$ the function $y \to -\rho_{a,c}(\mu,\nu;1/y)$ is completely monotonic and log-convex on $[0,\infty)$ for all $\mu, \nu > 0$ and there exists a nonnegative measure τ supported on $[0,\infty)$ such that*

$$\rho_{a,c}(\mu,\nu;x) = -\int_{[0,\infty)} e^{-t/x} d\tau(t).$$

Corollary 15 *Under hypotheses and notation of Theorem 5*

$$\frac{(a)_\nu(c)_\nu}{(a+\mu)_\nu(c+\mu)_\nu} \le \frac{h_{a,c}(0;x)h_{a,c}(\mu+\nu;x)}{h_{a,c}(\nu;x)h_{a,c}(\mu;x)} \le 1$$

for $\nu \in \mathbb{N}$, $x, \mu \ge 0$.

Combining corollaries 12 and 15 we obtain the following two-sided bounds for the Turánian:

$$\frac{h_0^2[(c+\nu)_\nu - (c)_\nu]}{\Gamma(c+\nu)\Gamma(c+2\nu)} \le h_{a,c}(\nu;x)^2 - h_{a,c}(0;x)h_{a,c}(2\nu;x) \le \left(1 - \frac{(a)_\nu(c)_\nu}{(a+\nu)_\nu(c+\nu)_\nu}\right) h_{a,c}(\nu;x)^2. \quad (23)$$

The bounds are valid for $\nu \in \mathbb{N}$, $a, c > 0$ and assuming that $\{h_n\}_{n\ge 0}$ is a nonnegative sequence which is also log-concave and without internal zeros if $a \in (0,c) \cup (c+1,\infty)$.

4. Applications and relation to other work. In this section we will demonstrate how Theorems 1 to 6 and their corollaries lead to various new inequalities for the Kummer, Gauss and generalized hypergeometric functions and their ratios and logarithmic derivatives.

Example 1. The first very natural candidate to apply the theory presented in the previous section is the Kummer function ${}_1F_1(a;c;x)$. Indeed, setting

$$f_{a,c}(\mu;x) = {}_1F_1(a+\mu;c+\mu;x), \qquad g_{a,c}(\mu;x) = \frac{\Gamma(a+\mu)}{\Gamma(c+\mu)} {}_1F_1(a+\mu;c+\mu;x),$$

$$h_{a,c}(\mu;x) = \frac{1}{\Gamma(c+\mu)} {}_1F_1(a+\mu;c+\mu;x) \ \text{ and } \ q_{a,c}(\mu;x) = \Gamma(a+\mu) {}_1F_1(a+\mu;c+\mu;x)$$

we obtain examples of functions defined by (11), (15), (19) and (21), respectively, and satisfying the corresponding theorems and corollaries. These facts extend and refine some previous results due to Baricz [3, Theorem 2] and the second author [15]. In particular, we obtain the following bounds for the Turánian:

$$\frac{2x(c-a)}{(c)_3} \le {}_1F_1(a+1;c+1;x)^2 - {}_1F_1(a;c;x){}_1F_1(a+2;c+2;x) \le \frac{c-a}{c(a+1)} {}_1F_1(a+1;c+1;x)^2 \quad (24)$$

if $c \ge a > 0$ and

$$\frac{a-c}{c(c+1)} \le \frac{a}{c} {}_1F_1(a+1;c+1;x)^2 - \frac{a+1}{c+1} {}_1F_1(a;c;x){}_1F_1(a+2;c+2;x) \le \frac{a-c}{c(c+1)} {}_1F_1(a+1;c+1;x)^2 \quad (25)$$

if $a \ge c > 0$. Simple rearrangements of the righthand side of (24) and the lefthand side of (25) give

$$\frac{a}{c} {}_1F_1(a+1;c+1;x)^2 - \frac{a+1}{c+1} {}_1F_1(a;c;x){}_1F_1(a+2;c+2;x) \begin{cases} \le 0, & c \ge a > 0, \\ \ge 0, & a \ge c > 0. \end{cases} \quad (26)$$

Substituting the contiguous relation

$${}_1F_1(a+2;c+2;x) = \frac{(c+1)(x-c)}{(a+1)x} {}_1F_1(a+1;c+1;x) + \frac{c(c+1)}{(a+1)x} {}_1F_1(a;c;x) \quad (27)$$

into (26) we get after some algebra:

$$xy^2 + (c-x)y - a \begin{cases} \le 0, & c \ge a > 0, \\ \ge 0, & a \ge c > 0. \end{cases}$$

where

$$y = \frac{{}_1F_1'(a;c;x)}{{}_1F_1(a;c;x)} = \frac{a\,{}_1F_1(a+1;c+1;x)}{c\,{}_1F_1(a;c;x)}.$$

In a similar fashion writing (26) with $a \to a+1$, $c \to c+1$ we obtain:

$$\frac{a+1}{c+1}{}_1F_1(a+2;c+2;x)^2 - \frac{a+2}{c+2}{}_1F_1(a+1;c+1;x){}_1F_1(a+3;c+3;x) \begin{cases} \leq 0, & c \geq a > 0, \\ \geq 0, & a \geq c > 0. \end{cases}$$

Using contiguous relation (27) twice this leads to

$$(ax+c)y^2 - a(x-c+1)y - a^2 \begin{cases} \geq 0, & c \geq a > 0, \\ \leq 0, & a \geq c > 0. \end{cases}$$

Solving the two quadratics we obtain

$$\frac{x-c+1+\sqrt{(x-c+1)^2+4ax+4c}}{2x+2c/a} \leq \frac{{}_1F_1'(a;c;x)}{{}_1F_1(a;c;x)} \leq \frac{x-c+\sqrt{(x-c)^2+4ax}}{2x}$$

if $c \geq a > 0$ and

$$\frac{x-c+\sqrt{(x-c)^2+4ax}}{2x} \leq \frac{{}_1F_1'(a;c;x)}{{}_1F_1(a;c;x)} \leq \frac{x-c+1+\sqrt{(x-c+1)^2+4ax+4c}}{2x+2c/a}$$

if $a \geq c > 0$. Note that for $a = c$ both bounds reduce to 1. It is also easy to check that both bounds give correct value 1 at $x = \infty$ and correct value a/c at $x = 0$. Moreover, the upper bound in the first inequality has a correct term of order $O(1/x)$ around infinity, while the lower bound has a correct term of order $O(x)$ around zero. Note that similar but different bounds have been obtained in our recent paper [14].

Integrating these bound from 0 to x we obtain

$$B_1(x) \leq {}_1F_1(a;c;x) \leq B_2(x) \text{ if } c \geq a > 0; \quad B_2(x) \leq {}_1F_1(a;c;x) \leq B_1(x) \text{ if } a \geq c > 0,$$

where (we set $b = (a+1)(a-c)$ for brevity)

$$B_1(x) = \frac{(2+2a)^{-b/a}c^{(a^2-b)/a}\left(1+2a-c+x+\sqrt{(x-c+1)^2+4ax+4c}\right)^{(a^2+b)/2a}}{\left(c^2(a+1)+a-c+(a^2+b)x+(a^2-b)\sqrt{(x-c+1)^2+4ax+4c}\right)^{(a^2-b)/2a}} \times$$

$$\times \exp\left\{\frac{x-c-1+\sqrt{(x-c+1)^2+4ax+4c}}{2}\right\}$$

$$B_2(x) = \frac{(4ac)^{c/2}}{(2a)^a}\frac{(2a+\sqrt{(x-c)^2+4ax}+x-c)^{a-c/2}}{(2ax/c+\sqrt{(x-c)^2+4ax}-(x-c))^{c/2}}\exp\left\{\frac{\sqrt{(x-c)^2+4ax}+x-c}{2}\right\}$$

All the above bounds can be easily extended to $x < 0$ using the Kummer identity ${}_1F_1(a;c;x) = e^x{}_1F_1(c-a;c;-x)$.

Example 2. The ratio

$$r(x) := \frac{{}_2F_1(a+1,b;c+1;x)}{{}_2F_1(a,b;c;x)}$$

was first developed into continued fraction by Euler. Later, Gauss found a different continued fraction which became more popular than the original fraction of Euler, see [2, paragraph 2.5] for details and references. Here we will derive bounds for this ratio under some restrictions on

parameters which are related in some way (see below) to the continued fraction of Euler. According to [2, formula (2.5.3)]

$$\frac{a+1}{c+1}{}_2F_1(a+2,b;c+2;x) = \frac{c+(a-b+1)x}{(c-b+1)x}{}_2F_1(a+1,b;c+1;x) - \frac{c}{(c-b+1)x}{}_2F_1(a,b;c;x). \quad (28)$$

Further, setting $g_n = (b)_n$ in (15) we get

$$g_{a,c}(\mu;x) = \frac{\Gamma(a+\mu)}{\Gamma(c+\mu)}{}_2F_1(a+\mu,b;c+\mu;x).$$

Then it follows from Corollary 5 (with $\mu=\nu=1$ and using $g_0>0$) that

$$\frac{a}{c}({}_2F_1(a+1,b;c+1;x))^2 \le \frac{a+1}{c+1}{}_2F_1(a,b;c;x){}_2F_1(a+2,b;c+2;x), \quad 0\le x<1, \qquad (29)$$

if $c\ge a>0$. Substituting (28) here we obtain

$$\frac{a}{c}({}_2F_1(a+1,b;c+1;x))^2 \le \frac{c+(a-b+1)x}{(c-b+1)x}{}_2F_1(a,b;c;x){}_2F_1(a+1,b;c+1;x) - \frac{c({}_2F_1(a,b;c;x))^2}{(c-b+1)x}$$

or, after division by $({}_2F_1(a,b;c;x))^2$,

$$\frac{a}{c}r(x)^2 - \frac{c+(a-b+1)x}{(c-b+1)x}r(x) + \frac{c}{(c-b+1)x} \le 0$$

Solving this quadratic inequality for $c-b+1<0$ and $c-b+1>0$ we arrive at

$$r(x) \le \frac{c+(a-b+1)x - \sqrt{(c+(a-b+1)x)^2 - 4a(c-b+1)x}}{2(a/c)(c-b+1)x}, \text{ if } c+1<b,$$

$$r(x) \ge \frac{c+(a-b+1)x - \sqrt{(c+(a-b+1)x)^2 - 4a(c-b+1)x}}{2(a/c)(c-b+1)x}, \text{ if } c+1>b.$$

Note that for $c=b+1$ both inequalities turn into correct equality $r(x)=c/(c-(c-a)x)$. It is also easy to verify that $r(0)=1$ coincides with the value of the bound at $x=0$. Using rather standard techniques the expression on the right of the two formulas above can be developed into continued fraction:

$$\frac{c+(a-b+1)x - \sqrt{(c+(a-b+1)x)^2 - 4a(c-b+1)x}}{2(a/c)(c-b+1)x} = \frac{c}{a(b-c-1)x}\mathop{\mathbf{K}}_{n=0}^{\infty}\frac{a(b-c-1)x}{c+(a-b+1)x}$$

which is interesting to compare with the continued fraction of Euler:

$$r(x) = \frac{c}{a(b-c)x}\mathop{\mathbf{K}}_{n=0}^{\infty}\frac{(a+n)(b-c-n)x}{c+n+(a-b+n+1)x}.$$

Here, we employed the usual notation

$$\mathop{\mathbf{K}}_{n=0}^{\infty}\frac{a_n}{b_n} = \cfrac{a_0}{b_0+\cfrac{a_1}{b_1+\cdots}}.$$

If the last fraction for $r(x)$ is made 1-periodic starting from $n=0$,

$$\frac{c}{a(b-c)x}\mathop{\mathbf{K}}_{n=0}^{\infty}\frac{a(b-c)x}{c+(a-b+1)x} = \frac{c+(a-b+1)x - \sqrt{(c+(a-b+1)x)^2 - 4a(c-b)x}}{2(a/c)(c-b)x},$$

we get an approximation which, by numerical tests, underestimates $r(x)$ and is less precise then our bound above. We can obtain a sequence of improving approximations to $r(x)$ by continued fractions which are 1-periodic starting from $n = N$, $N = 1, 2, \ldots$. Each approximation in this sequence is a rational function of x and square root of some quadratic of x and is easily computable.

We note the the above bounds can be extended to negative values of the argument by an application of Pfaff's transformation [2, formula (2.2.6)]

$$ {}_2F_1(a, b; c; x) = (1 - x)^{-a} {}_2F_1(a, c - b; c; x/(x - 1)). $$

Example 3. The application of Theorems 1 to 6 to generalized hypergeometric function is largely based on the following lemma.

Lemma 9 *Denote by $e_k(x_1, \ldots, x_q)$ the k-th elementary symmetric polynomial,*

$$ e_0(x_1, \ldots, x_q) = 1, \quad e_k(x_1, \ldots, x_q) = \sum_{1 \le j_1 < j_2 \cdots < j_k \le q} x_{j_1} x_{j_2} \cdots x_{j_k}, \quad k \ge 1. $$

Suppose $q \ge 1$ and $0 \le r \le q$ are integers, $a_i > 0$, $i = 1, \ldots, q - r$, $b_i > 0$, $i = 1, \ldots, q$, and

$$ \frac{e_q(b_1, \ldots, b_q)}{e_{q-r}(a_1, \ldots, a_{q-r})} \le \frac{e_{q-1}(b_1, \ldots, b_q)}{e_{q-r-1}(a_1, \ldots, a_{q-r})} \le \cdots \le \frac{e_{r+1}(b_1, \ldots, b_q)}{e_1(a_1, \ldots, a_{q-r})} \le e_r(b_1, \ldots, b_q). \tag{30} $$

Then the sequence of hypergeometric terms (if $r = q$ the numerator is 1),

$$ f_n = \frac{(a_1)_n \cdots (a_{q-r})_n}{(b_1)_n \cdots (b_q)_n}, $$

is log-concave, i.e. $f_{n-1} f_{n+1} \le f_n^2$, $n = 1, 2, \ldots$ It is strictly log-concave unless $r = 0$ and $a_i = b_i$, $i = 1, \ldots, q$.

The proof of this lemma for $r = 0$ follows from [10, Theorem 4.4]. For general r see [17, Lemma 2] and the last paragraph of that paper.

We note that it has been demonstrated in [16, Lemma 2] that (30) is true for $r = 0$ if majorization conditions

$$ \sum_{i=1}^{k} b_i \le \sum_{i=1}^{k} a_i \quad \text{for} \quad k = 1, 2, \ldots, q, $$

hold, where

$$ 0 < a_1 \le a_2 \le \cdots \le a_q, \quad 0 < b_1 \le b_2 \le \cdots \le b_q. $$

Consider the following functions ($p, q \ge 1$):

$$ f_{a,c}(\mu; x) = {}_pF_q(a + \mu, a_2, \ldots, a_p; c + \mu, c_2, \ldots, c_q; x), $$

$$ g_{a,c}(\mu; x) = \frac{\Gamma(a + \mu)}{\Gamma(c + \mu)} {}_pF_q(a + \mu, a_2, \ldots, a_p; c + \mu, c_2, \ldots, c_q; x), $$

$$ h_{a,c}(\mu; x) = \frac{1}{\Gamma(c + \mu)} {}_pF_q(a + \mu, a_2, \ldots, a_p; c + \mu, c_2, \ldots, c_q; x) $$

and

$$ q_{a,c}(\mu; x) = \Gamma(a + \mu) {}_pF_q(a + \mu, a_2, \ldots, a_p; c + \mu, c_2, \ldots, c_q; x). $$

Assuming all parameters are positive these functions satisfy Theorems 2, 3, 4(a), 5(a), 6 and their corollaries without any further restrictions. If, in addition, $p \le q$ and the vectors $(a_2, \ldots, a_p)$ and $(b_2, \ldots, b_q)$ satisfy Lemma 9 with $r = q - p$ then $f_{a,c}(\mu; x)$, $g_{a,c}(\mu; x)$ and $h_{a,c}(\mu; x)$ satisfy Theorems 1, 4(b) and 5(b), respectively. These facts imply a number of presumably new inequalities for the generalized hypergeometric function. In particular, if $\nu \in \mathbb{N}$, $x \ge 0$ and under conditions (30) the function $f_{a,c}(\mu; x)$ defined above satisfies (17) for $c \ge a > 0$, $g_{a,c}(\mu; x)$ satisfies (18) for $a \ge c > 0$ and $h_{a,c}(\mu; x)$ satisfies (23) for all $a, c > 0$.

5. Acknowledgements. We thank Sergei Sitnik for reading the manuscript and several useful remarks. The research was supported by the Ministry of Education and Science of Russia, projects 14.A18.21.0353 and 1.1773.2011, and the Russian Foundation for Basic Research, project 11-01-00038-a.

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